Sequences and Series Handout
CMSC 208
Spring 2020

Sequences

A sequence is an ordered list of numbers. The numbers in a sequence are called its **elements**.

We can describe sequences in several ways. One way is to list the elements of the sequence:

1, 2, 4, 8, 16, …

We use the notation "..." to show that the sequence continues following a pattern established by the previous elements in the list.

We can also describe sequences in terms of a starting value and a formula for calculating the next value. For example, the sequence which lists all positive even numbers in increasing order can be given by:

$a_0 = 0$
$a_n = 2*n$

This is sometimes called an "iterative" definition of the sequence. The notation a_i represents the ith element of the sequence. The value "i" is called the **index** of the element.

Some sequences can be defined in terms of a starting value and a formula for calculating each element from a previous element. This is a **recursive** or **inductive** definition of the sequence. For example, we can define the even number sequence recursively like this:

$a_0 = 0$
$a_n = a_{n-1} + 2$

Convergence

For many sequences, the terms grow larger and larger as the index "i" gets bigger. We say that these sequences **diverge** to infinity.

However, some sequences instead get closer and closer to some value. We say that these sequences **converge** to that value. For example, the sequence defined by:

$a_0 = 1$
$a_n = 1 / (n+1)$

which can be written 1, ½, ¼, ...

converges to **0**.

A third possibility is that a sequence oscillates between a small set of values – but we will not see very many of these **periodic sequences** in this class.

Series

A **numeric series** is the sum of the elements of a sequence. If the sequence is finite, we say the series is a **finite series**. If the sequence has an infinite number of terms, we say the series is an **infinite series**.

We can write a series as a set of terms added together, such as:

1 + ½ + ¼ + …

(Which adds up to 2)

Or

1 + 2 + 4 + 8

(Which adds up to 15)

Or we can use **summation notation**:

$$\sum_{i=0}^{3} 2^i$$

The way to read this notation is "the sum of 2^i as i goes from 0 to 3". The i=0 at the bottom tells us our starting value, the three at the top tells us our ending value and the 2^i gives us our formula.

$$\sum_{i=0}^{3} 2^i = 2^0 + 2^1 + 2^2 + 2^3 = 1 + 2 + 4 + 8 = 15.$$

Similarly, we can use summation notation for infinite series:

$$\sum_{i=1}^{\infty} \frac{1}{i}$$

The way to read this notation is "the sum of $\frac{1}{i}$ as i goes from 1 to infinity". The i=1 at the bottom tells us our starting value, the three at the top tells us our ending value and the $\frac{1}{i}$ gives us our formula.

If an infinite sequence converges or oscillates, this sum will be a finite value. If the sequence diverges, the sum will grow to infinity (either positive or negative infinity!).

Many series can also be written as a **closed form** solution in which we can describe the sum of the elements using a simple formula which contains no … or summation signs. For example, the series: $\sum_{1}^{n} i$ can be written as just $n\frac{(n-1)}{2}$.

Some important series formulas to know are:

$$\sum_{i=1}^{n} 1 \quad = \text{n.}$$

$$\sum_{i=1}^{n} c \quad = \mathbf{c}*\text{n for any constant } \mathbf{c}.$$

$$\sum_{i=a}^{b} (x+y) = \sum_{i=a}^{b} x + \sum_{i=a}^{b} y$$

$$\sum_{1}^{n} i \quad = \quad n\frac{(n-1)}{2}$$

$$\sum_{1}^{n} i^2 \quad = \quad \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} ar^i = a\frac{(1-r^n)}{1-r}$$ **if** -1 < r < 1 (this is called the Geometric Series formula)

If we let the geometric series go to infinity, this becomes:

$$\sum_{i=1}^{\infty} ar^i = a\frac{1}{1-r}$$

Another important series to know is the harmonic series:

$$\sum_{i=1}^{n} \frac{1}{i}$$

When n is infinity, this diverges (gets larger and larger), but for finite values of n, it approximates the natural logarithm of n.